

# DOUBLY COMMUTING SUBMODULES OF THE HARDY MODULE OVER POLYDISCS

JAYDEB SARKAR, AMOL SASANE, AND BRETT D. WICK<sup>‡</sup>

**ABSTRACT.** In this note we establish a vector-valued version of Beurling's Theorem (the Lax-Halmos Theorem) for the polydisc. As an application of the main result, we provide necessary and sufficient conditions for the completion problem in  $H^\infty(\mathbb{D}^n)$ .

## 1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In [2], Beurling described all the invariant subspaces for the operator  $M_z$  of “multiplication by  $z$ ” on the Hilbert space  $H^2(\mathbb{D})$  of the disc. In [4], Peter Lax extended Beurling's result to the (finite-dimensional) vector-valued case (while also considering the Hardy space of the half plane). Lax's vectorial case proof was further extended to infinite-dimensional vector spaces by Halmos, see [10]. The characterization of  $M_z$ -invariant subspaces obtained is the following famous result.

**THEOREM 1.1** (Beurling-Lax-Halmos). *Let  $\mathcal{S}$  be a closed nonzero subspace of  $H_{E_*}^2(\mathbb{D})$ . Then  $\mathcal{S}$  is invariant under multiplication by  $z$  if and only if there exists a Hilbert space  $E$  and an inner function  $\Theta \in H_{E \rightarrow E_*}^\infty(\mathbb{D})$  such that  $\mathcal{S} = \Theta H_E^2(\mathbb{D})$ .*

For  $n \in \mathbb{N}$  and  $E_*$  a Hilbert space,  $H_{E_*}^2(\mathbb{D}^n)$  is the set of all  $E_*$ -valued holomorphic functions in the polydisc  $\mathbb{D}^n$ , where  $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$  (with boundary  $\mathbb{T}$ ) such that

$$\|f\|_{H_{E_*}^2(\mathbb{D}^n)} := \sup_{0 < r < 1} \left( \int_{\mathbb{T}^n} \|f(r\mathbf{z})\|_{E_*}^2 d\mathbf{z} \right)^{1/2} < +\infty.$$

On the other hand, if  $\mathcal{L}(E, E_*)$  denotes the set of all continuous linear transformations from  $E$  to  $E_*$ , then  $H_{E \rightarrow E_*}^\infty(\mathbb{D}^n)$  denotes the set of all  $\mathcal{L}(E, E_*)$ -valued holomorphic functions with  $\|f\|_{H_{E \rightarrow E_*}^\infty(\mathbb{D}^n)} := \sup_{\mathbf{z} \in \mathbb{D}^n} \|f(\mathbf{z})\|_{\mathcal{L}(E, E_*)} < \infty$ .

A natural question is then to ask what happens in the case of several variables, for example when one considers the Hardy space  $H_{E_*}^2(\mathbb{D}^n)$  of the polydisc  $\mathbb{D}^n$ . It is known that in general, a Beurling-Lax-Halmos type characterization of subspaces of the Hardy Hilbert space is not possible [7]. It is however, easy to see that the Hardy space on the polydisc  $H_{E_*}^2(\mathbb{D}^n)$ , when  $n > 1$ , satisfies the *doubly commuting* property, that is, for all  $1 \leq i < j \leq n$

$$M_{z_i}^* M_{z_j} = M_{z_j} M_{z_i}^*.$$

---

1991 *Mathematics Subject Classification.* Primary 46J15; Secondary 47A15, 30H05, 47A56.

*Key words and phrases.* Invariant subspace, shift operator, Doubly commuting, Hardy algebra on the polydisc, Completion Problem.

<sup>‡</sup> Research supported in part by National Science Foundation DMS grants # 1001098 and # 955432.

We impose this additional assumption to the submodules of  $H_{E_*}^2(\mathbb{D}^n)$  and call that class of submodules as doubly commuting submodules. More precisely:

DEFINITION 1.2. *A commuting family of bounded linear operators  $\{T_1, \dots, T_n\}$  on some Hilbert space  $\mathcal{H}$  is said to be doubly commuting if*

$$T_i T_j^* = T_j^* T_i,$$

for all  $1 \leq i, j \leq n$  and  $i \neq j$ .

A closed subspace  $\mathcal{S}$  of  $H_E^2(\mathbb{D}^n)$  which is invariant under  $M_{z_1}, \dots, M_{z_n}$  is said to be a doubly commuting submodule if  $\mathcal{S}$  is a submodule, that is,  $M_{z_i} \mathcal{S} \subseteq \mathcal{S}$  for all  $i$  and the family of module multiplication operators  $\{R_{z_1}, \dots, R_{z_n}\}$  where

$$R_{z_i} := M_{z_i}|_{\mathcal{S}},$$

for all  $1 \leq i \leq n$ , is doubly commuting, that is,

$$R_{z_i} R_{z_j}^* = R_{z_j}^* R_{z_i},$$

for all  $i \neq j$  in  $\{1, \dots, n\}$ .

In this note we completely characterize the doubly commuting submodules of the vector-valued Hardy module  $H_{E_*}^2(\mathbb{D}^n)$  over the polydisc, and this is the content of our main theorem. This result is an analogue of the classical Beurling-Lax-Halmos Theorem on the Hardy space over the unit disc.

THEOREM 1.3. *Let  $\mathcal{S}$  be a closed nonzero subspace of  $H_{E_*}^2(\mathbb{D}^n)$ . Then  $\mathcal{S}$  is a doubly commuting submodule if and only if there exists a Hilbert space  $E$  with  $E \subseteq E_*$ , where the inclusion is up to unitary equivalence, and an inner function  $\Theta \in H_{E \rightarrow E_*}^\infty(\mathbb{D}^n)$  such that*

$$\mathcal{S} = M_\Theta H_E^2(\mathbb{D}^n).$$

In the special scalar case  $E_* = \mathbb{C}$  and when  $n = 2$  (the bidisc), this characterization was obtained by Mandrekar in [5], and the proof given there relies on the Wold decomposition for two variables [9]. Our proof is based on the more natural language of Hilbert modules and a generalization of Wold decomposition for doubly commuting isometries [8].

As an application of this theorem, we can establish a version of the Completion Property for the algebra  $H^\infty(\mathbb{D}^n)$ . Suppose that  $E \subset E_c$ . Recall that the Completion Problem for  $H^\infty(\mathbb{D}^n)$  is the problem of characterizing the functions  $f \in H_{E \rightarrow E_c}^\infty(\mathbb{D}^n)$  such that there exists an invertible function  $F \in H_{E_c \rightarrow E_c}^\infty(\mathbb{D}^n)$  with  $F|_E = f$ .

In the case of  $H^\infty(\mathbb{D})$ , the Completion Problem was settled by Tolokonnikov in [11]. In that paper, it was pointed out that there is a close connection between the Completion Problem and the characterization of invariant subspaces of  $H^2(\mathbb{D})$ . Using Theorem 1.3 we then have the following analogue of the results in [11].

THEOREM 1.4 (Tolokonnikov's Lemma for the Polydisc). *Let  $f \in H_{E \rightarrow E_c}^\infty(\mathbb{D}^n)$  with  $E \subset E_c$  and  $\dim E < \infty$ . Then the following statements are equivalent:*

- (i) *There exists a function  $g \in H_{E_c \rightarrow E}^\infty(\mathbb{D}^n)$  such that  $gf \equiv I$  in  $\mathbb{D}^n$  and the operators  $M_{z_1}, \dots, M_{z_n}$  doubly commute on the subspace  $\ker M_g$ .*

- (ii) *There exists a function  $F \in H_{E_c \rightarrow E_c}^\infty(\mathbb{D}^n)$  such that  $F|_E = f$ ,  $F|_{E_c \ominus E}$  is inner, and  $F^{-1} \in H_{E_c \rightarrow E_c}^\infty(\mathbb{D}^n)$ .*

In Section 2 we give a proof of Theorem 1.3, and subsequently, in Section 3, we use this theorem to study the Completion Problem for  $H^\infty(\mathbb{D}^n)$ , providing a proof of Theorem 1.4.

## 2. BEURLING-LAX-HALMOS THEOREM FOR THE POLYDISC

We begin by characterizing the “reducing submodules” of  $H_E^2(\mathbb{D}^n)$ . Recall that a closed subspace  $\mathcal{S} \subseteq H_E^2(\mathbb{D}^n)$  is said to be a *reducing submodule* of  $H_E^2(\mathbb{D}^n)$  if  $M_{z_i}\mathcal{S}$ ,  $M_{z_i}^*\mathcal{S} \subseteq \mathcal{S}$  for all  $i = 1, \dots, n$ .

**PROPOSITION 2.1.** *Let  $\mathcal{S}$  be a closed subspace of  $H_E^2(\mathbb{D}^n)$ . Then  $\mathcal{S}$  is a reducing submodule of  $H_E^2(\mathbb{D}^n)$  if and only if*

$$\mathcal{S} = H_{E_*}^2(\mathbb{D}^n),$$

for some closed subspace  $E_*$  of  $E$ .

*Proof.* Let  $\mathcal{S}$  be a reducing submodule of  $H_E^2(\mathbb{D}^n)$ , that is, for all  $1 \leq i \leq n$  we have

$$M_{z_i}P_{\mathcal{S}} = P_{\mathcal{S}}M_{z_i}.$$

Let

$$\mathbb{S}(z, w) = \prod_{j=1}^n (1 - \bar{w}_j z_j)^{-1},$$

be the Cauchy kernel on the polydisc  $\mathbb{D}^n$ . Now following Agler’s hereditary functional calculus [1], we have, with  $\mathbf{M}_{\mathbf{z}} := (M_{z_1}, \dots, M_{z_n})$  that

$$\begin{aligned} \mathbb{S}^{-1}(\mathbf{M}_{\mathbf{z}}, \mathbf{M}_{\mathbf{z}}) &= \left( \prod_{i=1}^n (1 - z_i \bar{w}_i) \right) (\mathbf{M}_{\mathbf{z}}, \mathbf{M}_{\mathbf{z}}) \\ &= \sum_{0 \leq i_1 < \dots < i_l \leq n} (-1)^l (z_{i_1} \cdots z_{i_l} \bar{w}_{i_1} \cdots \bar{w}_{i_l}) (\mathbf{M}_{\mathbf{z}}, \mathbf{M}_{\mathbf{z}}) \\ &= \sum_{0 \leq i_1 < \dots < i_l \leq n} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_l}} M_{z_{i_1}}^* \cdots M_{z_{i_l}}^*, \end{aligned}$$

and hence for all  $\mathbf{z}, \mathbf{w} \in \mathbb{D}^n$  and  $\eta, \zeta \in E$  we have

$$\begin{aligned}
& \langle \mathbb{S}^{-1}(\mathbf{M}_{\mathbf{z}}, \mathbf{M}_{\mathbf{z}}) \mathbb{S}(\cdot, \mathbf{z})\eta, \mathbb{S}(\cdot, \mathbf{w})\zeta \rangle \\
&= \left\langle \sum_{0 \leq i_1 < \dots < i_l \leq n} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_l}} M_{z_{i_1}}^* \cdots M_{z_{i_l}}^* \mathbb{S}(\cdot, \mathbf{z})\eta, \mathbb{S}(\cdot, \mathbf{w})\zeta \right\rangle \\
&= \sum_{0 \leq i_1 < \dots < i_l \leq n} (-1)^l \left\langle M_{z_{i_1}}^* \cdots M_{z_{i_l}}^* \mathbb{S}(\cdot, \mathbf{z})\eta, M_{z_{i_1}}^* \cdots M_{z_{i_l}}^* \mathbb{S}(\cdot, \mathbf{w})\zeta \right\rangle \\
&= \sum_{0 \leq i_1 < \dots < i_l \leq n} (-1)^l \bar{z}_{i_1} \cdots \bar{z}_{i_l} w_{i_1} \cdots w_{i_l} \langle \mathbb{S}(\cdot, \mathbf{z}), \mathbb{S}(\cdot, \mathbf{w}) \rangle \langle \eta, \zeta \rangle \\
&= \mathbb{S}^{-1}(\mathbf{w}, \mathbf{z}) \mathbb{S}(\mathbf{w}, \mathbf{z}) \langle \eta, \zeta \rangle \\
&= \langle \eta, \zeta \rangle \\
&= \langle P_E \mathbb{S}(\cdot, \mathbf{z})\eta, \mathbb{S}(\cdot, \mathbf{w})\zeta \rangle
\end{aligned}$$

where  $P_E$  denotes the orthogonal projection of  $H_E^2(\mathbb{D}^n)$  onto the space of all constant functions. Since  $\{\mathbb{S}(\cdot, \mathbf{z})\eta : \mathbf{z} \in \mathbb{D}^n, \eta \in E\}$  is a total subset of  $H_E^2(\mathbb{D}^n)$ , we have that

$$\mathbb{S}^{-1}(\mathbf{M}_{\mathbf{z}}, \mathbf{M}_{\mathbf{z}}) = P_E.$$

Consequently,

$$P_E P_S = \mathbb{S}^{-1}(\mathbf{M}_{\mathbf{z}}, \mathbf{M}_{\mathbf{z}}) P_S = P_S \mathbb{S}^{-1}(\mathbf{M}_{\mathbf{z}}, \mathbf{M}_{\mathbf{z}}) = P_S P_E.$$

Therefore,  $P_S P_E$  is an orthogonal projection and

$$P_S P_E = P_E P_S = P_{E_*},$$

where  $E_* := E \cap \mathcal{S}$ . Hence, for any

$$f = \sum_{\mathbf{k} \in \mathbb{N}^n} a_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} \in \mathcal{S},$$

where  $a_{\mathbf{k}} \in E$  for all  $\mathbf{k} \in \mathbb{N}^n$ , we have

$$f = P_S f = P_S \left( \sum_{\mathbf{k} \in \mathbb{N}^n} M_{\mathbf{z}}^{\mathbf{k}} a_{\mathbf{k}} \right) = \sum_{\mathbf{k} \in \mathbb{N}^n} M_{\mathbf{z}}^{\mathbf{k}} P_S a_{\mathbf{k}}.$$

But  $P_S a_{\mathbf{k}} = P_S P_E a_{\mathbf{k}} \in E_*$ . Consequently,  $M_{\mathbf{z}}^{\mathbf{k}} P_S a_{\mathbf{k}} \in H_{E_*}^2(\mathbb{D}^n)$  for all  $\mathbf{k} \in \mathbb{N}^n$  and hence  $f \in H_{E_*}^2(\mathbb{D}^n)$ . That is,  $\mathcal{S} \subseteq H_{E_*}^2(\mathbb{D}^n)$ . For the reverse inclusion, it is enough to observe that  $E_* \subseteq \mathcal{S}$  and that  $\mathcal{S}$  is a reducing submodule. The converse part is immediate. Hence the lemma follows.  $\blacksquare$

Now let  $\mathcal{S}$  be a doubly commuting submodule of  $H_E^2(\mathbb{D}^n)$ . Then

$$R_{z_i} R_{z_i}^* = M_{z_i} P_S M_{z_i}^* P_S = M_{z_i} P_S M_{z_i}^*,$$

implies that  $R_{z_i} R_{z_i}^*$  is an orthogonal projection of  $\mathcal{S}$  onto  $z_i \mathcal{S}$  and hence  $I_{\mathcal{S}} - R_{z_i} R_{z_i}^*$  is an orthogonal projection of  $\mathcal{S}$  onto  $\mathcal{S} \ominus z_i \mathcal{S}$ , that is,

$$I_{\mathcal{S}} - R_{z_i} R_{z_i}^* = P_{\mathcal{S} \ominus z_i \mathcal{S}},$$

for all  $i = 1, \dots, n$ . Define  $\mathcal{W}_i = \text{ran}(I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)$  for all  $i = 1, \dots, n$ , and

$$\mathcal{W} = \bigcap_{i=1}^n \mathcal{W}_i.$$

It readily follows by doubly commutativity of  $\mathcal{S}$  that

$$\mathcal{W} = \text{ran}\left(\prod_{i=1}^n (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)\right).$$

Now we present a wandering subspace theorem concerning doubly commuting submodules of  $H_E^2(\mathbb{D}^n)$ . The result is a consequence of a several variables analogue of the classical Wold decomposition theorem as obtained by Gaspar and Suciú [3]. We provide a direct proof (also see Corollary 3.2 in [8]).

**THEOREM 2.2.** *Let  $\mathcal{S}$  be a doubly commuting submodule of  $H_E^2(\mathbb{D}^n)$ . Then*

$$\mathcal{S} = \sum_{\mathbf{k} \in \mathbb{N}^n} \oplus z^{\mathbf{k}} \mathcal{W}.$$

*Proof.* First, note that if  $\mathcal{S}$  is a submodule of  $H_E^2(\mathbb{D}^n)$  then  $R_{z_i}$  is a shift, that is, the unitary part of the Wold decomposition of  $R_{z_i}$  is trivial:

$$\bigcap_{k \in \mathbb{N}} R_{z_i}^{*k} \mathcal{S} = \{0\},$$

for each  $i = 1, \dots, n$ . Moreover, if  $\mathcal{S}$  is doubly commuting then

$$R_{z_i}(I_{\mathcal{S}} - R_{z_j} R_{z_j}^*) = (I_{\mathcal{S}} - R_{z_j} R_{z_j}^*) R_{z_i},$$

for all  $i \neq j$ . Therefore  $\mathcal{W}_j$  is a  $R_{z_i}$ -reducing subspace for all  $i \neq j$ . Note also that for all  $1 \leq m < n$ ,

$$\begin{aligned} \bigcap_{i=1}^{m+1} \mathcal{W}_i &= \text{ran}\left(\prod_{i=1}^{m+1} (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)\right) \\ &= \text{ran}\left(\prod_{i=1}^m (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) - R_{z_{m+1}} R_{z_{m+1}}^* \prod_{i=1}^m (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*)\right) \\ &= \text{ran}\left(\prod_{i=1}^m (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) - R_{z_{m+1}} \prod_{i=1}^m (I_{\mathcal{S}} - R_{z_i} R_{z_i}^*) R_{z_{m+1}}^*\right) \\ &= (\mathcal{W}_1 \cap \dots \cap \mathcal{W}_m) \ominus z_{m+1}(\mathcal{W}_1 \cap \dots \cap \mathcal{W}_m), \end{aligned}$$

and hence

$$(\mathcal{W}_1 \cap \dots \cap \mathcal{W}_m) \ominus z_{m+1}(\mathcal{W}_1 \cap \dots \cap \mathcal{W}_m) = \bigcap_{i=1}^{m+1} \mathcal{W}_i.$$

We use mathematical induction to prove that for all  $2 \leq m \leq n$ , we have

$$\mathcal{S} = \sum_{\mathbf{k} \in \mathbb{N}^m} \oplus z^{\mathbf{k}} (\mathcal{W}_1 \cap \dots \cap \mathcal{W}_m).$$

First, by Wold decomposition theorem for the shift  $R_{z_1}$  on  $\mathcal{S}$  we have

$$\mathcal{S} = \sum_{k_1 \in \mathbb{N}} \oplus R_{z_1}^{k_1} \mathcal{W}_1 = \sum_{k_1 \in \mathbb{N}} \oplus z_1^{k_1} \mathcal{W}_1.$$

Again by applying Wold decomposition for  $R_{z_2}|_{\mathcal{W}_1} \in \mathcal{L}(\mathcal{W}_1)$  we have

$$\mathcal{W}_1 = \sum_{k_2 \in \mathbb{N}} \oplus R_{z_2}^{k_2} (\mathcal{W}_1 \ominus z_2 \mathcal{W}_1) = \sum_{k_2 \in \mathbb{N}} \oplus z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2),$$

and hence

$$\mathcal{S} = \sum_{k_1 \in \mathbb{N}} \oplus z_1^{k_1} \left( \sum_{k_2 \in \mathbb{N}} \oplus z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2) \right) = \sum_{k_1, k_2 \in \mathbb{N}} \oplus z_1^{k_1} z_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2).$$

Finally, let

$$\mathcal{S} = \sum_{\mathbf{k} \in \mathbb{N}^m} \oplus z^{\mathbf{k}} (\mathcal{W}_1 \cap \dots \cap \mathcal{W}_m),$$

for some  $m < n$ . Then we again apply the Wold decomposition on the isometry

$$R_{z_{m+1}}|_{\mathcal{W}_1 \cap \dots \cap \mathcal{W}_m} \in \mathcal{L}(\mathcal{W}_1 \cap \dots \cap \mathcal{W}_m)$$

to obtain

$$\begin{aligned} \mathcal{W}_1 \cap \dots \cap \mathcal{W}_m &= \sum_{k_{m+1} \in \mathbb{N}} \oplus z_{m+1}^{k_{m+1}} \left( (\mathcal{W}_1 \cap \dots \cap \mathcal{W}_m) \ominus z_{m+1} \mathcal{W}_1 \cap \dots \cap \mathcal{W}_m \right) \\ &= \sum_{k_{m+1} \in \mathbb{N}} \oplus z_{m+1}^{k_{m+1}} (\mathcal{W}_1 \cap \dots \cap \mathcal{W}_m \cap \mathcal{W}_{m+1}), \end{aligned}$$

which yields

$$\mathcal{S} = \sum_{\mathbf{k} \in \mathbb{N}^{m+1}} \oplus z^{\mathbf{k}} (\mathcal{W}_1 \cap \dots \cap \mathcal{W}_{m+1}).$$

This completes the proof. ■

We now turn to the proof of Theorem 1.3.

*Proof of Theorem 1.3.* By Theorem 2.2 we have

$$\mathcal{S} = \sum_{\mathbf{k} \in \mathbb{N}^n} \oplus z^{\mathbf{k}} \left( \bigcap_{i=1}^n \mathcal{W}_i \right).$$

Now define the Hilbert space  $E$  by

$$E = \bigcap_{i=1}^n \mathcal{W}_i,$$

and the linear operator  $V : H_E^2(\mathbb{D}^n) \rightarrow H_{E_*}^2(\mathbb{D}^n)$  by

$$V \left( \sum_{\mathbf{k} \in \mathbb{N}^n} a_{\mathbf{k}} z^{\mathbf{k}} \right) = \sum_{\mathbf{k} \in \mathbb{N}^n} M_z^{\mathbf{k}} a_{\mathbf{k}},$$

where

$$\sum_{\mathbf{k} \in \mathbb{N}^n} a_{\mathbf{k}} z^{\mathbf{k}} \in H_E^2(\mathbb{D}^n)$$

and  $a_{\mathbf{k}} \in E$  for all  $\mathbf{k} \in \mathbb{N}^n$ . It is evident that  $V \in \mathcal{L}(H_E^2(\mathbb{D}^n), H_{E_*}^2(\mathbb{D}^n))$  is isometric module map onto  $\mathcal{S}$ . Therefore,

$$V = M_{\Theta},$$

for some inner function  $\Theta \in H_{E \rightarrow E_*}^{\infty}(\mathbb{D}^n)$ . Finally, that  $\dim E \leq \dim E_*$  follows from the boundary behavior of the  $H^{\infty}$ -inner functions on the polydisc.

To prove the converse part, let  $\mathcal{S} = M_{\Theta} H_E^2(\mathbb{D}^n)$  be a submodule of  $H_{E_*}^2(\mathbb{D}^n)$  for some inner function  $\Theta \in H_{E \rightarrow E_*}^{\infty}(\mathbb{D}^n)$ . Then

$$P_{\mathcal{S}} = M_{\Theta} M_{\Theta}^*,$$

and hence for all  $i \neq j$ ,

$$\begin{aligned} M_{z_i} P_{\mathcal{S}} M_{z_j}^* &= M_{z_i} M_{\Theta} M_{\Theta}^* M_{z_j}^* = M_{\Theta} M_{z_i} M_{z_j}^* M_{\Theta}^* = M_{\Theta} M_{z_j}^* M_{z_i} M_{\Theta}^* \\ &= M_{\Theta} M_{z_j}^* M_{\Theta}^* M_{\Theta} M_{z_i} M_{\Theta}^* = M_{\Theta} M_{\Theta}^* M_{z_j}^* M_{z_i} M_{\Theta} M_{\Theta}^* \\ &= P_{\mathcal{S}} M_{z_j}^* M_{z_i} P_{\mathcal{S}}. \end{aligned}$$

This implies

$$R_{z_j}^* R_{z_i} = P_{\mathcal{S}} M_{z_j}^* P_{\mathcal{S}} M_{z_i} |_{\mathcal{S}} = P_{\mathcal{S}} M_{z_j}^* M_{z_i} |_{\mathcal{S}} = M_{z_i} P_{\mathcal{S}} M_{z_j}^* = R_{z_i} R_{z_j}^*,$$

that is,  $\mathcal{S}$  is a doubly commuting submodule. This completes the proof. ■

### 3. TOLOKONNIKOV'S LEMMA FOR THE POLYDISC

We will need the following lemma, which is a polydisc version of a similar result proved in the case of the disc in Nikolski's book [6, p.44-45]. Here we use the notation  $M_g$  for the multiplication operator on  $H_E^2$  induced by  $g \in H_{E \rightarrow E_*}^{\infty}$ .

**LEMMA 3.1** (Lemma on Local Rank). *Let  $E, E_c$  be Hilbert spaces, with  $\dim E < \infty$ . Let  $g \in H_{E_c \rightarrow E}^{\infty}(\mathbb{D}^n)$  be such that*

$$\ker M_g = \{h \in H_{E_c}^2(\mathbb{D}^n) : g(z)h(z) \equiv 0\} = \Theta H_{E_a}^2(\mathbb{D}^n),$$

where  $E_a$  is a Hilbert space and  $\Theta$  is a  $\mathcal{L}(E_a, E_c)$ -valued inner function. Then

$$\dim E_c = \dim E_a + \text{rank } g,$$

where  $\text{rank } g := \max_{\zeta \in \mathbb{D}^n} \text{rank } g(\zeta)$ .

*Proof.* We have  $\ker M_g = \{h \in H_{E_c}^2(\mathbb{D}^n) : gh \equiv 0\}$ . If  $\zeta \in \mathbb{D}^n$ , then let

$$[\ker M_g](\zeta) := \{h(\zeta) : h \in \ker M_g\}.$$

It is easy to check that  $[\ker M_g](\zeta) = \Theta(\zeta)E_a$ . If  $\dim E_c = \infty$ , then one can show that  $\dim[\ker M_g](\zeta) = \infty$ . So  $\dim E_a = \infty$  as well, and this proves the claim.

So we assume that  $\dim E_c < \infty$ . It is clear that for  $\zeta \in \mathbb{D}^n$ ,

$$\dim \Theta(\zeta)E_a = \dim[\ker M_g](\zeta) \leq \dim \ker g(\zeta) = \dim E_c - \text{rank } g(\zeta).$$

From the analyticity of  $\Theta$  and  $g$ , it follows that there exists a point  $\zeta_1 \in \mathbb{D}^n$ , with

$$\dim E_a = \dim \Theta(\zeta_1)E_a, \quad \text{rank } g(\zeta_1) = \text{rank } g.$$

Hence  $\dim E_a \leq \dim E_c - \text{rank } g$ .

For the proof of the opposite inequality, let us consider a principal minor  $g_1(\zeta_1)$  of the matrix of the operator  $g(\zeta_1)$  (with respect to two arbitrary fixed bases in  $E_c$  and  $E$  respectively). Then  $\det g_1 \in H^\infty$ ,  $\det g_1 \neq 0$ . Let  $E_c = E_{c,1} \oplus E_{c,2}$ ,  $E = E_1 \oplus E_2$  ( $\dim E_{c,1} = \dim E_1 = \text{rank } g(\zeta_1)$ ) be the decompositions of the spaces  $E_c$  and  $E$  corresponding to this minor, and let

$$g(\zeta) = \begin{bmatrix} g_1(\zeta) & g_2(\zeta) \\ \gamma_1(\zeta) & \gamma_2(\zeta) \end{bmatrix}, \quad \zeta \in \mathbb{D}^n,$$

be the matrix representation of  $g(\zeta)$  with respect to this decomposition. Using

$$\gamma_2 \det g_1 = \gamma_1 g_1^{\text{co}} g_2, \quad \text{where } g_1^{\text{co}} := (\det g_1) g_1^{-1},$$

we get the inclusion  $M_\Omega H_{E_{c,2}}^2(\mathbb{D}^n) \subset \ker M_g$ , where  $\Omega \in H_{E_{c,2} \rightarrow E_c}^\infty(\mathbb{D}^n)$  is given by

$$\Omega = \begin{bmatrix} g_1^{\text{co}} g_2 \\ -\det g_1 \end{bmatrix}.$$

We have  $\text{rank } \Omega = \dim E_{c,2} = \dim E_c - \text{rank } g = \dim \ker(g(\zeta_1))$ . Consequently, we obtain  $\dim[\ker M_g](\zeta_1) \geq \dim \ker(g(\zeta_1))$ .  $\blacksquare$

We now turn to the extension of Tolokonnikov's Lemma to the polydisc.

*Proof of Theorem 1.4.* (ii)  $\Rightarrow$  (i): If  $g := P_E F^{-1}$ , then  $gf = I$ . It only remains to show that the operators  $M_{z_1}, \dots, M_{z_n}$  are doubly commuting on the space  $\ker M_g$ . Let  $\Theta, \Gamma$  be such that:

$$F = \begin{bmatrix} f & \Theta \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} g \\ \Gamma \end{bmatrix}.$$

Since  $FF^{-1} = I_{E_c}$ , it follows that  $fg + \Theta\Gamma = I_{E_c}$ . Thus if  $h \in H_{E_c}^2(\mathbb{D}^n)$  is such that  $gh = 0$ , then  $\Theta(\Gamma h) = h$ , and so  $h \in \Theta H_{E_c \ominus E}^2(\mathbb{D}^n)$ . Hence  $\ker M_g \subset \text{ran } M_\Theta$ . Also, since  $F^{-1}F = I$ , it follows that  $g\Theta = 0$ , and so  $\text{ran } M_\Theta \subset \ker M_g$ . So  $\ker M_g = \text{ran } M_\Theta = \Theta H_{E_c \ominus E}^2(\mathbb{D}^2)$ . By Theorem 1.3, the operators  $M_{z_1}, \dots, M_{z_n}$  must doubly commute on the subspace  $\ker M_g$ .

(i)  $\Rightarrow$  (ii): Let

$$\mathcal{S} := \{h \in H_{E_c}^2(\mathbb{D}^n) : g(z)h(z) \equiv 0\} = \ker g.$$

$\mathcal{S}$  is a closed non-zero invariant subspace of  $H_{E_c}^2(\mathbb{D}^n)$ . Also, by assumption,  $M_{z_1}, \dots, M_{z_n}$  are doubly commuting operators on  $\mathcal{S}$ . Then by the above Theorem 1.3, there exists an auxiliary Hilbert space  $E_a$  and an inner function  $\tilde{\Theta}$  with values in  $\mathcal{L}(E_a, E_c)$  with  $\dim E_a \leq \dim E_c$  such that

$$\mathcal{S} = \tilde{\Theta} H_{E_a}^2(\mathbb{D}^n).$$

By the Lemma on Local Rank,  $\dim E_a = \dim E_c - \text{rank } g = \dim E_c - \dim E = \dim(E_c \ominus E)$ . Let  $U$  be a (constant) unitary operator from  $E_c \ominus E$  to  $E_a$  and define  $\Theta := \tilde{\Theta}U$ . Then  $\Theta$  is



inner, and we have that  $\ker g = \Theta H_{E_c \ominus E}^2(\mathbb{D}^n)$ . To get  $F \in H_{E_c \rightarrow E_c}^\infty(\mathbb{D}^n)$  define the function  $F$  for  $z \in \mathbb{D}^n$  by

$$F(z)e := \begin{cases} f(z)e & \text{if } e \in E \\ \Theta(z)e & \text{if } e \in E_c \ominus E. \end{cases}$$

We note that  $F \in H^\infty(\mathbb{D}^n)$  and  $F|_E = f$ . We now show that  $F$  is invertible. With this in mind, we first observe that

$$(I - fg)H_{E_c}^2(\mathbb{D}^n) \subset \Theta H_{E_c \ominus E}^2(\mathbb{D}^n) = \ker M_g.$$

This follows since  $g(I - fg)h = gh - gh = 0$  for all  $h \in H_{E_c}^2(\mathbb{D}^n)$ . Thus we have that  $\Theta^*(I - fg) \in H_{E_c \rightarrow E_c \ominus E}^\infty(\mathbb{D}^n)$ . Now, define  $\Omega = g \oplus \Theta^*(I - fg)$ . Clearly  $\Omega \in H_{E_c \rightarrow E_c}^\infty(\mathbb{D}^n)$ . Next, note that

$$F\Omega = fg + \Theta\Theta^*(I - fg) = I.$$

Similarly,

$$\begin{aligned} \Omega F &= gf\mathbb{P}_E + \Theta^*(I - fg)(f\mathbb{P}_E + \Theta\mathbb{P}_{E_c \ominus E}) \\ &= \mathbb{P}_E + \Theta^*(f\mathbb{P}_E - fgf\mathbb{P}_E + \Theta\mathbb{P}_{E_c \ominus E}) \\ &= \mathbb{P}_E + \Theta^*\Theta\mathbb{P}_{E_c \ominus E} = I. \end{aligned}$$

Thus we have that  $F^{-1} \in H^\infty(\mathbb{D}^n; E_c \rightarrow E_c)$ . ■

**REMARK 3.2.** *Theorem 1.4 for the polydisc is different from Tolokonnikov's lemma in the disc in which one does not demand that the completion  $F$  has the property that  $F|_{E_c \ominus E}$  is inner. But, from the proof of Tolokonnikov's lemma in the case of the disc (see [6]), one can see that the following statements are equivalent for  $f \in H_{E \rightarrow E_c}^\infty(\mathbb{D})$  with  $E \subset E_c$  and  $\dim E < \infty$ :*

- (i) *There exists a function  $g \in H_{E_c \rightarrow E}^\infty(\mathbb{D})$  such that  $gf \equiv I$  in  $\mathbb{D}^n$ .*
- (ii) *There exists a function  $F \in H_{E_c \rightarrow E_c}^\infty(\mathbb{D})$  such that  $F|_E = f$ , and  $F^{-1} \in H_{E_c \rightarrow E_c}^\infty(\mathbb{D})$ .*
- (ii') *There exists a function  $F \in H_{E_c \rightarrow E_c}^\infty(\mathbb{D})$  such that  $F|_E = f$ ,  $F|_{E_c \ominus E}$  is inner, and  $F^{-1} \in H_{E_c \rightarrow E_c}^\infty(\mathbb{D})$ .*

## REFERENCES

- [1] Jim Agler, *The Arveson extension theorem and coanalytic models*, Integral Equations and Operator Theory **5** (1982), 608–631.
- [2] Arne Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1948), 17.
- [3] Dumitru Gaşpar and Nicolae Suciu, *Wold decompositions for commutative families of isometries*, An. Univ. Timişoara Ser. Ştiinţ. Mat. **27** (1989), no. 2, 31–38.
- [4] Peter D. Lax, *Translation invariant spaces*, Acta Math. **101** (1959), 163–178.
- [5] V. Mandrekar, *The validity of Beurling theorems in polydiscs*, Proc. Amer. Math. Soc. **103** (1988), no. 1, 145–148.
- [6] N. K. Nikol'skiĭ, *Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer-Verlag, Berlin, 1986. Spectral function theory; With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller; Translated from the Russian by Jaak Peetre.
- [7] Walter Rudin, *Function theory in polydiscs*, W. A. Benjamin, Inc., New York-Amsterdam, 1969.
- [8] Jaydeb Sarkar, *Wold decomposition for doubly commuting isometries*, preprint, arXiv:1304.7454.

- [9] Marek Słociński, *On the Wold-type decomposition of a pair of commuting isometries*, Ann. Polon. Math. **37** (1980), no. 3, 255–262.
- [10] Béla Sz.-Nagy and Ciprian Foiaş, *Harmonic analysis of operators on Hilbert space*, Translated from the French and revised, North-Holland Publishing Co., Amsterdam, 1970.
- [11] Vadim Tolokonnikov, *Extension problem to an invertible matrix*, Proc. Amer. Math. Soc. **117** (1993), no. 4, 1023–1030.

J. SARKAR, INDIAN STATISTICAL INSTITUTE, STATISTICS AND MATHEMATICS UNIT, 8TH MILE, MYSORE ROAD, BANGALORE, 560059, INDIA

*E-mail address:* jay@isibang.ac.in, jaydeb@gmail.com

*URL:* <http://www.isibang.ac.in/~jay/>

A. SASANE, MATHEMATICS DEPARTMENT, LONDON SCHOOL OF ECONOMICS, HOUGHTON STREET, LONDON WC2A 2AE, U.K.

*E-mail address:* sasane@lse.ac.uk

BRETT D. WICK, SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, 686 CHERRY STREET, ATLANTA, GA USA 30332-0160, U.S.A.

*E-mail address:* wick@math.gatech.edu

*URL:* [www.math.gatech.edu/~wick](http://www.math.gatech.edu/~wick)